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Anisotropic Voter Model

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A majority vote model subject to anisotropic voting rules is studied in two dimensions using a first-order mean-field approximation and Monte Carlo simulations. The critical behavior is consistent with the 2D Ising universality class.

KEY WORDS: Stochastic spin systems; majority vote models; anisotropy.

1. INTRODUCTION

Polling models are probabilistic lattice models for the evolution of opinion (yes or no) on some issue. They are useful toy models for the study of non-equilibrium systems, since they display nontrivial phase transitions between stationary phases but have rather simple stochastic microscopically irreversible rules.[1,2]

Grinstein *et al.*[3] argued, back in 1985, that stochastic spin-flip models with two states per site and updating rules of a short-range nature with up/down symmetry should belong to the (kinetic) Ising model universality class. Their argument rests on the stability of the dynamic Ising fixed point in $d=4-\varepsilon$ dimensions with respect to perturbations preserving both the spin inversion and the lattice symmetries. This hypothesis has received extensive confirmation from MC simulations[4-8,19,16] as well as from analytic calculations.[10-12] The models investigated include Ising models with a competition of two (or three[19]) Glauber-like rates at different temperatures[10,12,6,7] or a combination of spin-flip and spin-exchange

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dynamics[20] and other types of transition rules with the restrictions mentioned above.[8,4] Ising behavior has also been found in systems with three states per site under certain kinds of rules.[9] The critical behavior of a two-dimensional nonequilibrium model with two-valued bonds and different bond strengths in the x and y directions has been studied by Blöte *et al.*[16] with no clear evidence of a distinct universality class.

Another approach to the study of stochastic nonequilibrium systems was taken by Domany,[17] who established a mapping of some d-dimensional CA to $(d+1)$-dimensional equilibrium models with constrained coupling constants. The critical behavior of the former is related to the critical behavior of the latter on a disorder variety; however, due to the special nature of the transition of the equilibrium model, this information is usually not very illuminating.[21,18]

In the present work, we investigate the robustness of the Ising behavior with respect to anisotropy by considering an anisotropic version of one of the simplest nonequilibrium models, the majority vote model.

2. MODEL AND CALCULATIONS

The (isotropic) majority vote model is defined by a set of "spin" variables $\{\sigma_i\}$ taking value $+1$ or -1 on the sites of a d-dimensional hypercubic lattice, evolving by single-spin-flip dynamics with a flip probability w_i which depends on the state of nearest neighbors $\sigma \equiv \{\sigma_{i+\delta}\}$:

$$w_i(\sigma) = \frac{1}{2}\left[1 - \sigma_i(1-2q)\, S\left(\sum_\delta \sigma_{i+\delta}\right)\right] \tag{1}$$

where $S(x) = \mathrm{sign}(x)$ if $x \neq 0$ and $S(x) = 0$ if $x = 0$. The noise parameter q is the probability of aligning antiparallel to (disagreeing with) the majority of neighbors. In one dimension the model is exactly solvable: the stationary state is paramagnetic (P) for $0 < q < 1$, ferromagnetic (F) for $q = 0$, and antiferromagnetic (AF) for $q = 1$. In two dimensions MC calculations[4] have found an F phase for $0 \leqslant q \leqslant q_c$ (with $q_c = 0.075$ for a square lattice), an AF phase for $1 - q_c \leqslant q \leqslant 1$, and a P state for $q_c < q < 1 - q_c$; the critical behavior is 2D Ising.

We consider now anisotropic flip rates, namely

$$w_i(\sigma) = x w_i^1(\sigma) + (1-x)\, w_i^2(\sigma) \tag{2}$$

where $w_i^1(\sigma)$ and $w_i^2(\sigma)$ are, respectively, the 1D and 2D versions of (1) and x is the anisotropy parameter ($0 \leqslant x \leqslant 1$). Now, with probability x the voter only looks at left and right neighbors and is influenced by four neighbors with probability $1-x$. Rates (2) are expected to reduce the F phase to $0 \leqslant q \leqslant q_c(x)$ with $q_c(0) = 0.075$ and $q_c(1) = 0$. Since the AF phase follows by symmetry, we will consider only values of q less than 1/2.

The time evolution of expectation values is given by

$$\frac{d\langle \sigma_i \rangle}{dt} = -2\langle \sigma_i w_i(\sigma) \rangle \tag{3}$$

$$\frac{d\langle \sigma_i \sigma_j \rangle}{dt} = -2\langle \sigma_i \sigma_j (w_i(\sigma) + w_j(\sigma)) \rangle \tag{4}$$

We looked for stationary solutions of these equations in two dimensions using a mean-field approximation (pair approximation); this study is complemented by a numerical (MC) simulation.

2.1. Pair Approximation

In this approximation we need only the single- and two-particle probabilities, which can, in turn, be written as functions of the magnetization $m \equiv \langle \sigma_0 \rangle$ and the nearest-neighbor pair correlations $r_1 \equiv \langle \sigma_0 \sigma_1 \rangle$ and $r_2 \equiv \langle \sigma_0 \sigma_2 \rangle$ (for horizontal and vertical bonds, respectively). The probability of a cluster formed by a central spin σ_0 and its four neighbors is then

$$P(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = P(\sigma_0) \prod_{j=1}^{4} \frac{P(\sigma_0, \sigma_j)}{P(\sigma_0)}$$

where $P(1, 1) = (1 + 2m + r_k)/4$, $P(-1, -1) = (1 - 2m + r_k)/4$ and $P(1, -1) = P(-1, 1) = (1 - r_k)/4$ with $r_k = r_1$ (r_2) for horizontal (vertical) bonds and

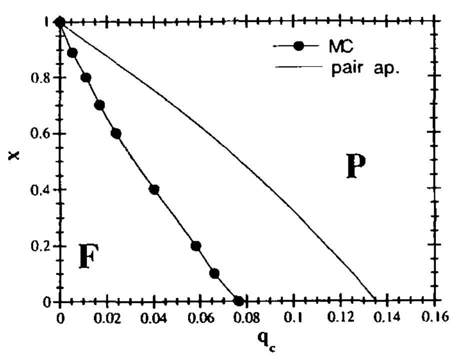

Fig. 1. Phase diagram obtained by a pair approximation and by MC simulation, showing the paramagnetic (P) and ferromagnetic (F) states.

$P(1) = (1+m)/2$ and $P(-1) = (1-m)/2$ as usual. The stationary solutions of Eqs. (3) and (4) in the pair approximation are of the form

$$m = mF_1(x, q; r_1, r_2) + O(m^3)$$
$$r_1 = F_2(x, q; r_1, r_2) + O(m^2)$$
$$r_2 = F_3(x, q; r_1, r_2) + O(m^2)$$

The F/P transition line, $q_c(x)$, is given by $F_1 = 1$ with $r_1 = F_2$ and $r_2 = F_3$ and is plotted in Fig. 1.

2.2. Monte Carlo Calculations

Monte Carlo simulations were performed on a square lattice of L^2 spins ($L \leqslant 92$) with periodic boundary conditions and random initial configurations. The simulation procedure and data analysis are similar to the ones described in refs. 4, 8, and 9. For given values of x and q, a set of configurations were generated by repeatedly choosing a spin at random and flipping with probability given by (2). Up to 300,000 MCS were taken (1 MCS equals L^2 spin-flip attempts); as we were interested in steady-state expectation values, a variable number of initial MCS (typically 150,000) were discarded prior to the averaging procedure.

For several system sizes, the magnetization $M_L \equiv \langle |m| \rangle = (1/L^2) \langle |\sum_i \sigma_i| \rangle$, fourth-order cumulant $U_L \equiv 1 - \langle m^4 \rangle / 3 \langle m^2 \rangle^2$, and susceptibility $\chi_L \equiv L^2(\langle m^2 \rangle - \langle |m| \rangle^2)$ were computed as a function of q for given anisotropy x (Figs. 2 and 3). The (infinite) system transition point $q_c(x)$ shown in Fig. 1 was obtained from the abscissa of the intersection of the curves $U_L(q)$ for various sizes L. We found that the value $U^* \equiv U_L(q_c) = 0.61 \pm 0.01$ is the same as in the equilibrium isotropic case.[13]

In order to investigate the critical behavior, we have studied more carefully the vicinity of the critical point and performed finite-size scaling analysis[14,15] for anisotropy values $x = 0$ and $x = 0.2$. The size dependences of χ_L and M_L at the critical point

$$\chi_L(q_c) = AL^{\gamma/\nu}$$
$$M_L(q_c) = BL^{-\beta/\nu}$$

yield the exponents γ/ν and β/ν; the maximum value of the susceptibility also scales as $L^{\gamma/\nu}$. The log–log plots of Figs. 4 and 5 confirm scaling with exponents consistent with Onsager's values $\gamma/\nu = 1.75$ and $\beta/\nu = 0.125$. On the other hand, the value for which χ_L has a maximum, $q_c(L)$, is expected to be a linear function of $L^{-1/\nu}$,

$$q_c(L) = q_c + CL^{-1/\nu}$$

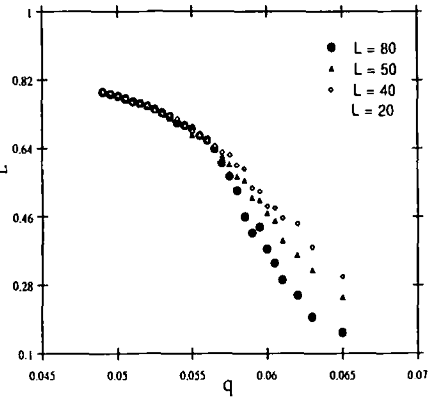

Fig. 2. Magnetization M_L as a function of q for various system sizes and anisotropy $x = 0.2$.

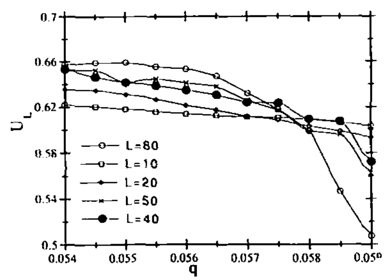

Fig. 3. Fourth-order cumulant U_L as a function of q for various system sizes and anisotropy $x = 0.2$.

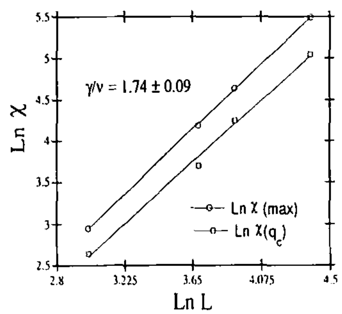

Fig. 4. Log–log plots of the maximum value of the susceptibility, $\chi_L(\max)$, and of the susceptibility at the critical point, $\chi_L(q_c)$, versus L for anisotropy $x = 0.2$. The slopes of the best fits yield $\gamma/\nu = 1.74 \pm 0.09$.

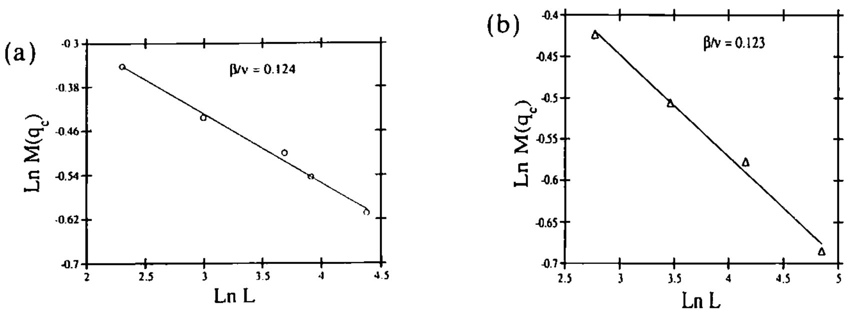

Fig. 5. (a) Log–log plot of the magnetization at the critical point versus system size for $x = 0.2$. The slope of the best fit yields $\beta/\nu = 0.124$. (b) The same, for the isotropic ($x = 0$) case.

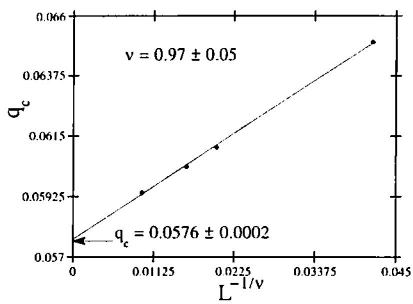

Fig. 6. Value of q at $\chi_L(\max)$ versus $L^{-1/\nu}$ for $\nu = 0.97$ and $x = 0.2$; the intersection with the vertical axis gives the critical point $q_c(0.2) = 0.0576 \pm 0.0002$.

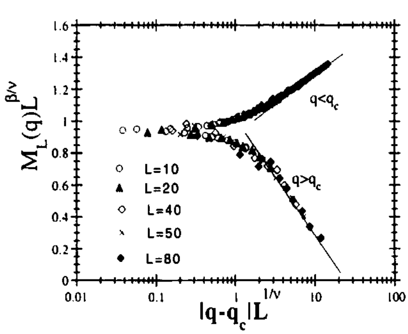

Fig. 7. Data-collapsing plot: $M_L(q)\, L^{\beta/\nu}$ as a function of $\log(|q - q_c|\, L^{1/\nu})$ for $q_c = 0.0578$, $\beta/\nu = 0.124$, $\nu = 0.96$; anisotropy $x = 0.2$.

Figure 6 shows that for $x = 0.2$ the best fit to a straight line is obtained taking $\nu = 0.97$; the intersection with the vertical axis gives an independent estimate of the critical value which agrees with the value $q_c = 0.0578$ obtained from the fourth-order cumulant. Finally we have performed a data-collapse plot of $M_L(q)\, L^{\beta/\nu}$ versus $\log(|q - q_c|\, L^{1/\nu})$ (Fig. 7) obtaining a reasonable fit over two decades of the horizontal variable when the above-mentioned values of exponents are used. For higher values of the anisotropy the critical region becomes rather narrow and the order parameter changes rapidly near the transition, making it very difficult to get good statistics. We have, however, checked that for $x = 0.6$ a reasonable fit to the data is obtained using Onsager exponents in the scaling analysis.

3. DISCUSSION AND CONCLUSION

We have studied a kind of dimensional crossover in stationary nonequilibrium phases of a majority vote model induced by anisotropic dynamic rules. The general behavior determined by Eqs. (1) and (2) is easily understood: for $x = 1$ each spin receives only two inputs from neighbors and that is not enough to sustain a ferromagnetic phase over a finite range of parameter space[21]; as soon as $x \neq 1$, the spin has nonzero probability of getting four inputs and the system may reach a stationary state with 2D ferromagnetic order. The situation is analogous (but *not equivalent*!) to a square lattice Ising model with a coupling $J_x = J$ in the x direction and bond dilution ($J_{iy} = J, 0$ with probability p, $1 - p$) in the y axis. In this case, a crossover from 1D to 2D is induced by the anisotropic dilution and the 2D Ising fixed point is stable with respect to this perturbation.[22]

Our MC study reproduces the results of ref. 4 when $x = 0$, namely 2D Ising critical behavior. Evidence for the same universality class is also presented for the anisotropic $x = 0.2$ case and qualitatively also for higher values of x, although it proved increasingly difficult to get reliable data as one approaches the first-order critical point at $q = 1$, $x = 1$. Our results indicate that 2D Ising critical behavior is stable with respect to this kind of anisotropy, therefore supporting universality for this type of nonequilibrium system. A similar conclusion was reached by Blöte and coworkers[16] in their study of a nonequilibrium model with a preferred direction.

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